Hermann Weyl's Space-Time Geometry

and

the Origin of Gauge Theory 100 years ago

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Weyl's Attempt to Unify Gravitation and Electromagnetism

Weyl to Einstein (March 1918):

> *"These days I succeeded, as I believe, to derive electricity and gravitation from a common source ...."*

Weyl's Generalization of Riemannian Geometry

Weyl's starting point was purely mathematical:

> *"But in Riemannian geometry described above there is contained a last element of geometry "at a distance" (ferngeometrisches Element) — with no good reason, as far as I can see; it is due only to the accidental development of Riemannian geometry from Euclidean geometry. The metric allows the two magnitudes of two vectors to be compared, not only at the same point, but at any arbitrarily separated points. A true infinitesimal geometry should, however, recognize only a principle for transferring the magnitude of a vector to an infinitesimally close point and then, on transfer to an arbitrary distant point, the integrability of the magnitude of a vector is no more to be expected than the integrability of its direction."*

Weyl's **physical speculation**:

"*On the removal of this inconsistency there appears a geometry that, surprisingly, when applied to the world, explains not only the gravitational phenomena but also the electrical. According to the resultant theory both spring from the same source, indeed in general one cannot separate gravitation and electromagnetism in a unique manner. In this theory all physical quantities have a world geometrical meaning; the action appears from the beginning as a pure number. It leads to an essentially unique universal law; it even allows us to understand in a certain sense why the world is four-dimensional*."

Weyl-Geometry:

• Spacetime manifold M is equipped with a *conformal structure* $[g]$:

$$g_1 \sim g_2 \quad \Leftrightarrow g_1(x) = e^{2\lambda(x)} g_2(x);$$

same null cones. (Lengths can only be compared at one and the same world point.)

• M has (torsion-free) linear connection $\to$ parallel transport, covariant derivative ∇.

• Linear connection respects conformal structure:

$$\nabla g = -2A \otimes g \qquad (\nabla_\alpha g_{\mu\nu} = -2A_\alpha g_{\mu\nu}), \qquad g \in [g], \tag{1}$$

where the map $A : [g] \to \Lambda^1(M)$ satisfies

$$A(e^{2\lambda} g) = A(g) - d\lambda. \tag{2}$$

$A(g)$is the *gauge potential* belonging to g, and (2) is what Weyl called a gauge transformation. In bundle theoretical language: A *Weyl connection* is a torsion-free connection of the conformal principle bundle (Weyl-bundle). [$\rightarrow$ (1), (2)]

Existence of covariant Weyl derivatives

Generalize the well-known Koszul treatment of the Levi-Civita connection. In particular we generalize the Koszul formula to

$$g(\nabla_Z Y, X) = g(\nabla_Z^{LC} Y, X) + [-A(X)g(Y,Z) + A(Y)g(Z,X) + A(Z)g(X,Y)].$$

Defines ∇_X in terms of g and A.

Derivation. Explicitly a Weyl connection satisfies

$$(\nabla_X g)(Y,Z) = Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = -2A(X)g(Y,Z).$$

Since the torsion vanishes, i.e., $\nabla_X Y - \nabla_Y X - [X,Y] = 0$

$$Xg(Y,Z) = g(\nabla_Y X, Z) + g([X,Y],Z) + g(Y, \nabla_X Z) + 2A(X)g(Y,Z).$$

Result follows after cyclic permutations.

With routine calculations one verifies that the generalized Koszul formula defines a covariant derivative with vanishing torsion, and moreover it satisfies defining property.

Local formula. Chose $X = \partial_i$, $Y = \partial_j$, , $Z = \partial_k$ of local coordinates:

$$\langle \nabla_{\partial_i} \partial_j, \partial_k \rangle = \frac{1}{2}(-g_{ji,k} + g_{ik,j} + g_{kj,i)}) + (-A_i g_{jk} + A_j g_{ki} + A_k g_{ij}).$$

Has to perform in the Christoffel symbols of the Levi-Civita connection the substitution

$$g_{ij,k} \to g_{ij,k} - 2A_k g_{ij}.$$

Non-integrable scale factor (gauge factor)
Ratio of lengths in q and p (measured with $g \in [g]$) *depends in general on the connecting path* γ:

$$\boxed{l(q) = \exp\left(-\int_\gamma A\right)\; l(p).}$$

This equation holds for all $g \in [g]$! Transport of length path-independent
$\Leftrightarrow$

$$F = dA = 0 \qquad (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0).$$

[Weyl geometry and the Weyl-Cartan Theorem ?]

Consider G-structures on a manifolds M: Reductions of the bundle of linear frames $L(M)$ to a subgroup $G \subset GL(n;\mathbb{R}$. For $G = \mathcal{O}(n)$, a G-structure is equivalent to a Riemannian metric on M.

The Weyl-Cartan Theorem states that for a *closed* subgroup $G \subset GL(n;\mathbb{R}$, $n \geq 3$ the following two conditions are equivalent:

(1) G is the subgroup of all matrices which preserve a certain non-degenerate quadratic form of any signature.

(2) For every n-dimensional manifold M and every reduced sub-bundle P of $L(M)$ with group G, there is a *unique* torsion-free connection in P.

Remark: Weyl-connections are not counter examples!

Change of calibration (gauge transformation)

∇ has <u>absolute</u> meaning. Passing to another calibration with metric $\bar{g}$, related to g by

$$\bar{g} = e^{2\lambda} g,$$

the potential A will also change to $\bar{A}$: by definition

$$\nabla \bar{g} = -2\bar{A} \otimes \bar{g};$$

on the other hand:

$$\nabla \bar{g} = \nabla(e^{2\lambda} g) = 2d\lambda \otimes \bar{g} + e^{2\lambda} \nabla g = 2d\lambda \otimes \bar{g} - 2A \otimes \bar{g}.$$

Thus

$$\bar{A} = A - d\lambda \quad (\bar{A}_\mu = A_\mu - \partial_\mu \lambda).$$

Hence: change of calibration of the metric induces a "gauge transformation" for A:

$$\boxed{g_{\mu\nu} \to e^{2\lambda} g_{\mu\nu}, \qquad A_\mu \to A_\mu - \partial_\mu \lambda.}$$

Only gauge classes have an absolute meaning. (The Weyl connection is, however, gauge-invariant; conceptually clear, can be verified by direct calculation.)

Electromagnetism and Gravitation

• Weyl assumes that his "purely infinitesimal geometry" describes the structure of spacetime

$\longrightarrow$

• physical laws should satisfy a double-invariance: 1. They must be invariant with respect to arbitrary smooth coordinate transformations. 2. They must be *gauge invariant*.

• **Interpretation**: $g_{\mu\nu}$ = gravitational potential; A_μ = vector potential, $F_{\mu\nu}$ = field strength of electromagnetism.

In the absence of electromagnetic fields ($F_{\mu\nu} = 0$) the scale factor $\exp(-\int_\gamma A)$ for length transport becomes path independent (integrable) and one can find a gauge such that A_μ vanishes for simply connected spacetime regions. In this special case one is in the same situation as in GR.

Action: generally invariant and gauge invariant. In his first paper Weyl proposes what we call nowadays the Yang-Mills action

$$S(g, A) = -\frac{1}{4}\int Tr(\Omega \wedge *\Omega).$$

Ω = curvature form, $*\Omega$ = Hodge dual*. Note that the latter is gauge invariant, i.e., independent of the choice of $g \in [g]$. In Weyl's geometry the curvature form splits: $\Omega = \hat{\Omega} + F$, $\hat{\Omega}$ = 'metric' piece. Correspondingly, the action also splits,

$$Tr(\Omega \wedge *\Omega) = Tr(\hat{\Omega} \wedge *\hat{\Omega}) + F \wedge *F;$$

second term = Maxwell action. Weyl's theory thus contains formally all aspects of a non-Abelian gauge theory.

Note: Einstein-Hilbert action is **not gauge invariant**.

*The integrand is in local coordinates indeed just the expression $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\sqrt{-g}dx^0 \wedge \ldots \wedge dx^3$ which is used by Weyl ($R_{\alpha\beta\gamma\delta}$= the curvature tensor of the Weyl connection).

19 year old **Wolfgang Pauli** analyzes the implications of Weyl's action (perihelion motion, light deflection).

Hermann Weyl to Pauli, 10 May, 1919:

> *"I am extremely pleased to be able to welcome you as a collaborator. However, it is almost inconceivable to me how you could possibly have succeeded at so young an age to get hold of all the means of knowledge and to acquire the liberty of thought that is needed to assimilate the theory of relativity."*

Independent of the precise form of the action Weyl shows that in his theory gauge invariance implies the conservation of electric charge in much the same way as general coordinate invariance leads to the 'conservation' of energy and momentum. This beautiful connection pleased him particularly:

> "*. . . [it] seems to me to be the strongest general argument in favour of the present theory — insofar as it is permissible to talk of justification in the context of pure speculation.*"

The invariance principles imply five 'Bianchi type' identities. Correspondingly, the five 'conservation laws' follow in two independent ways from the coupled field equations and may be "termed the eliminants" of the latter. These structural connections hold also in modern gauge theories.

Einstein's Objection and Reactions of other Physicists

If the idea of a non-integrable length connection (scale factor) is correct, then the behavior of clocks would depend on their history; in clear contradiction with empirical evidence, in particular with the *existence of stable atomic spectra*. Einstein therefore concludes:

> " ... *(if) one drops the connection of the ds to the measurement of distance and time, then relativity looses all its empirical basis*."

Illustration: static situation, only electric field: $A = \phi dt$, ϕ : t-independent; clock in fixed position, rate $l = l_0 \exp\left(\int_0^t \phi dt'\right) = l_0 e^{t\phi}$; consider two clocks C_1, C_2: C_2 stays in 2 for time t; (rate of C_2)/(rate of C_1) $= e^{-[\phi(2)-\phi(1)]t}$ $\rightarrow$ no spectral lines of definite frequencies.

> Einstein: "*it is impossible that the theory corresponds to nature.*"

Weyl to Einstein (10. 12. 1918)

"This [insistence] irritates me of course, because experience has proven that one can rely on your intuition; so unconvincing as your counter-arguments seem to me, as I have to admit [. . .]. By the way, you should not believe that I was driven to introduce the linear differential form in addition to the quadratic one by physical reasons. I wanted, just to the contrary, to get rid of this 'methodological inconsistency (Inkonsequenz)' which has been a bone of contention to me already much earlier. And then, to my surprise, I realized that it looked as if it might explain electricity. You clap your hands above your head and shout: But physics is not made this way !"

London (1927):

"In the face of such elementary experimental evidence, it must have been an unusually strong metaphysical conviction that prevented Weyl from abandoning the idea that Nature would have to make use of the beautiful geometrical possibility that was offered. He stuck to his conviction and evaded discussion of the above-mentioned contradictions through a rather unclear re-interpretation of the concept of "real state", which, however, robbed his theory of its immediate physical meaning and attraction."

Quote from a remarkable paper by London in which he suggested a reinterpretation of Weyl's principle of gauge invariance within the new quantum mechanics: The role of the metric is taken over by the wave function, and the rescaling of the metric has to be replaced by a phase change of the wave function.

Further early contributions: **Schrödinger (1922), Fock (1926), Klein (1926)**. F. London to Schrödinger (1926):

> *"Do you know a certain Herr Schrödinger who described, in the year 1922, a 'noteworthy property of quantum orbits'? Do you know this man? [...] Will you now immediately confess that, like a priest, you held the truth in your hands and kept it secret? [...]."*

Weyl's 1929 Classic:
"Electron and Gravitation"

Weyl in 1955 (Selecta):

> *"Later the quantum-theory introduced the Schrödinger-Dirac potential ψ of the electron-positron field; it carried with it an experimentally-based principle of gauge-invariance which guaranteed the conservation of charge, and connected the ψ with the electromagnetic potentials A_μ in the same way that my speculative theory had connected the gravitational potentials $g_{\mu\nu}$ with the A_μ, and measured the A_μ in known atomic, rather than unknown cosmological units. I have no doubt but that the correct context for the principle of gauge-invariance is here and not, as I believed in 1918, in the intertwining of electromagnetism and gravity."*

The richness and scope of the paper is clearly visible from the following table of contents:

“ *Introduction. Relationship of General Relativity to the quantum-theoretical field equations of the spinning electron: mass, gauge-invariance, distant-parallelism. Expected modifications of the Dirac theory. -I. Two-component theory: the wave function ψ has only two components. -§1. Connection between the transformation of the ψ and the transformation of a normal tetrad in four-dimensional space. Asymmetry of past and future, of left and right. -§2. In General Relativity the metric at a given point is determined by a normal tetrad. Components of vectors relative to the tetrad and coordinates. Covariant differentiation of ψ. -§3. Generally invariant form of the Dirac action, characteristic for the* *wave-field of matter. -§4. The differential conservation law of energy and momentum and the symmetry of the energy-momentum tensor as a consequence of the double-invariance (1) with respect to coordinate transformations (2) with respect to rotation of the tetrad. Momentum and moment of momentum for matter. -§5. Einstein's classical theory of gravitation in the new analytic formulation. Gravitational energy. -§6. The electromagnetic field. From the arbitrariness of the gauge-factor in ψ appears the necessity of introducing the electromagnetic potential. Gauge invariance and charge conservation. The space-integral of charge. The introduction of mass. Discussion and rejection of another possibility in which electromagnetism appears, not as an accompanying phenomenon of matter, but of gravitation.*"

from the **Introduction**:

"The Dirac field-equations for ψ together with the Maxwell equations for the four potentials A_μ of the electromagnetic field have an invariance property which is formally similar to the one which I called gauge-invariance in my 1918 theory of gravitation and electromagnetism; the equations remain invariant when one makes the simultaneous substitutions [...]."

"*The connection of this "gauge invariance" to the conservation of electric charge remains untouched. But a fundamental difference, which is important to obtain agreement with observation, is that the exponent of the factor multiplying ψ is not real but pure imaginary. ψ now plays the role that Einstein's ds played before. It seems to me that this new principle of gauge-invariance, which follows not from speculation but from experiment, tells us that the electromagnetic field is a necessary accompanying phenomenon, not of gravitation, but of the material wave-field represented by ψ. Since gauge-invariance involves an arbitrary function λ it has the character of "general" relativity and can naturally only be understood in that context.*"

Weyl's incorporation of the Dirac theory into GR

Weyl's reaction to Einstein's recent teleparallel unification attempt, and remarks by Wigner and others on possible connection with spin theory of the electron:

> *"I prefer not to believe in distant parallelism for a number of reasons. First my mathematical intuition objects to accepting such an artificial geometry; I find it difficult to understand the force that would keep the local tetrads at different points and in rotated positions in a rigid relationship. There are, I believe, two important physical reasons as well. The loosening of the rigid relationship between the tetrads at different points converts the gauge-factor $e^{i\lambda}$, which remains arbitrary with respect to ψ, from a constant to an arbitrary function of space-time. In other words, only through the loosening the rigidity does the established gauge-invariance become understandable."*

Tetrad Formalism

tetrad (Vierbein): basis of orthonormal vector fields $\{e_\alpha(x); \alpha = 0, 1, 2, 3\}$; $\{e^\alpha(x)\}$: dual basis of 1-forms; metric:

$$g = \eta_{\mu\nu} e^\mu(x) \otimes e^\nu(x), \qquad (\eta_{\mu\nu}) = diag(1, -1, -1, -1);$$

invariant under local Lorentz transformations:

$$e^\alpha(x) \to \Lambda^\alpha{}_\beta(x) e^\beta(x);$$

connection forms $\omega = (\omega^\alpha{}_\beta)$ have values in the Lie algebra of the homogeneous Lorentz group:

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0;$$

determined (in terms of the tetrad) by the first structure equation of Cartan:

$$de^\alpha + \omega^\alpha{}_\beta \wedge e^\beta = 0.$$

Connection forms (gauge potentials of gravity) transform in the same way as the gauge potential of a non-Abelian gauge theory:

$$\omega(x) \to \Lambda(x)\omega(x)\Lambda^{-1}(x) - d\Lambda(x)\Lambda^{-1}(x).$$

Field strengths = curvature forms $\Omega = (\Omega^{\mu}{}_{\nu})$; obtained from ω in exactly the same way as the Yang-Mills field strength from the gauge potential:

$$\Omega = d\omega + \omega \wedge \omega$$

(second structure equation).

Covariant derivative:

1. for vector field V^α:

$$DV^\alpha = dV^\alpha + \omega^\alpha{}_\beta V^\beta;$$

2. for Dirac field ψ:

$$D\psi = d\psi + \frac{1}{4}\omega_{\alpha\beta}\sigma^{\alpha\beta}\psi; \qquad \sigma^{\alpha\beta} = \frac{1}{2}[\gamma^\alpha, \gamma^\beta].$$

Einstein-Dirac system in massless case:

$$\mathcal{L} = \frac{1}{16\pi G}R - i\bar{\psi}\gamma^\mu D_\mu \psi.$$

Weyl discusses the two symmetries:

(i) local Lorentz invariance: $\Rightarrow$ symmetry of $T^{\mu\nu}$;

(ii) general coordinate invariance: $\Rightarrow \nabla_\nu T^{\mu\nu} = 0$.

The New Form of the Gauge-Principle

All this is a kind of a preparation for the final section of Weyl's paper, which has the title "electric field". Weyl says:

> " *We come now to the critical part of the theory. In my opinion the origin and necessity for the electromagnetic field is in the following. The components ψ_1 ψ_2 are, in fact, not uniquely determined by the tetrad but only to the extent that they can still be multiplied by an arbitrary "gauge-factor" $e^{i\lambda}$. The transformation of the ψ induced by a rotation of the tetrad is determined only up to such a factor. In special relativity one must regard this gauge-factor as a constant because here we have only a single point-independent tetrad. Not so in general relativity; every point has its own tetrad and hence its own arbitrary gauge-factor; because by the removal of the rigid connection between tetrads at different points the gauge-factor necessarily becomes an arbitrary function of position.*"

In this manner Weyl arrives at the gauge-principle in its modern form and emphasizes: “From the arbitrariness of the gauge-factor in ψ appears the necessity of introducing the electromagnetic potential.” The nonintegrable scale factor of the old theory is now replaced by a phase factor:

$$\exp\left(-\int_\gamma A\right) \to \exp\left(-i\int_\gamma A\right);$$

corresponds to the replacement of the original gauge group $\mathbb{R}$ by the compact group $U(1)$.

Einstein objection $\to$ Aharonov-Bohm effect!

Current conservation follows again in two independent ways:

1. from gauge inv. + field eqs. for matter;

2. from gauge inv. + field eqs. for elm. field;

corresponds to an identity in the coupled system of field equations which has to exist as a result of gauge invariance.

When Pauli saw the full version of Weyl's paper he became more friendly and wrote:

> *"In contrast to the nasty things I said, the essential part of my last letter has since been overtaken, particularly by your paper in Z. f. Physik. For this reason I have afterward even regretted that I wrote to you. After studying your paper I believe that I have really understood what you wanted to do (this was not the case in respect of the little note in the Proc.Nat.Acad.). First let me emphasize that side of the matter concerning which I am in full agreement with you: your incorporation of spinor theory into gravitational theory. I am as dissatisfied as you are with distant parallelism and your proposal to let the tetrads rotate independently at different space-points is a true solution."*

In brackets Pauli adds:

" Here I must admit your ability in Physics. Your earlier theory with $g'_{ik} = \lambda g_{ik}$ was pure mathematics and unphysical. Einstein was justified in criticizing and scolding. Now the hour of your revenge has arrived."

Then he remarks in connection with the mass-problem:

" Your method is valid even for the massive [Dirac] *case. I thereby come to the other side of the matter, namely the unsolved difficulties of the Dirac theory (two signs of m_0) and the question of the 2-component theory. In my opinion these problems will not be solved by gravitation . . . the gravitational effects will always be much too small."*

Earlier, on July 1 (1929), after Pauli had seen a preliminary account of Weyl's work, he wrote to him with familiar bluntness:

> *" Before me lies the April edition of the Proc.Nat.Acad. (US). Not only does it contain an article from you under "Physics" but shows that you are now in a 'Physical Laboratory': from what I hear you have even been given a chair in 'Physics' in America. I admire your courage; since the conclusion is inevitable that you wish to be judged, not for success in pure mathematics, but for your true but unhappy love for physics."*

Weyl on Pauli (1945):

"The mathematicians feel near to Pauli since he is distinguished among physicists by his highly developed organ for mathematics. Even so, he is a physicist; for he has to a high degree what makes the physicist; the genuine interest in the experimental facts in all their puzzling complexity. His accurate, instructive estimate of the relative weight of relevant experimental facts has been an unfailing guide for him in his theoretical investigations. Pauli combines in an exemplary way physical insight and mathematical skill."

Gauge invariance and quantization (Heisenberg and Pauli, 1928):
Maxwell Lagrangian $\rightarrow$

$$\pi_\mu = \frac{\partial L}{\partial(\partial_0 A_\mu)} = F_{0\mu}, \quad \Rightarrow \pi_0 \equiv 0, \quad \pi_i = F_{0i} = -E_i;$$

Gauss constraint $\nabla \cdot \mathbf{E} = 0$ in contradiction with the canonical commutation relations:

$$[A_j(\mathbf{x}), \pi_k(\mathbf{x}')] = i\delta^3(\mathbf{x} - \mathbf{x}')\delta_{jk}.$$

In fall 1928 Heisenberg proposed to proceed as follows: Add the term $-\frac{1}{2}\varepsilon(\partial_\mu A^\mu)^2$ to the Lagrangian; $\pi_0 = -\varepsilon\partial_\mu A^\mu$, canonical quantization scheme can be applied; take $\varepsilon \rightarrow 0$ at the end of all calculations.

Heisenberg and Pauli II (1930): Lorentz condition cannot be imposed as an operator identity but only as a supplementary condition selecting admissable states. ($\leftrightarrow$ Fermi, 1929).

Pauli turned to literature. In a letter of 18 February 1929 he wrote from Zürich to Oskar Klein:

> *"For my proper amusement I then made a short sketch of a utopian novel which was supposed to have the title 'Gulivers journey to Urania' and was intended as a political satire in the style of Swift against present-day democracy. [...] Caught in such dreams, suddenly in January, news from Heisenberg reached me that he is able, with the aid of a trick ... to get rid of the formal difficulties that stood against the execution of our quantum electrodynamics."*

On Pauli's invention of non-Abelian Kaluza-Klein Theory in 1953

There are documents which show that Wolfgang Pauli constructed in 1953 the first consistent generalization of the five-dimensional theory of Kaluza, Klein, Fock and others to a higher dimensional internal space. Because he saw no way to give masses to the gauge bosons, he refrained from publishing his results formally. This is still a largely unknown chapter of the early history of non-Abelian gauge and Kaluza-Klein theories.

At the Lorentz-Kammerlingh Onnes conference in Leiden (22-27 June 1953) Pauli asked Pais after his talk:

> *"...I would like to ask in this connection whether the transformation group with constant phases can be amplified in a way analogous to the gauge group for electromagnetic potentials in such a way that the meson-nucleon interaction is connected with the amplified group..."*

Pauli worked on the problem, and wrote on July 25, 1953 a long technical letter to Pais, with the motto: ”Ad *usum Delfini* only”. This letter begins with a personal part in which Pauli says that ”*the whole note for you is of course written in order to drive you further into the real virgin-country*”. The note has the interesting title:

“Written down July 22-25 1953, in order to see how it looks. Meson-Nucleon Interaction and Differential Geometry.”

Pauli generalizes the original Kaluza-Klein theory to a six-dimensional space and arrives through dimensional reduction at the essentials of an $SU(2)$ gauge theory. The extra-dimensions form a two-sphere S^2 with space-time dependent metrics on which the $SU(2)$ operates in a space-time-dependent manner. Pauli emphasizes that this transformation group *"seems to me therefore the natural generalization of the gauge-group* in case of a two-dimensional spherical surface". He then develops in 'local language' the geometry of what we now call a fibre bundle with a homogeneous space as typical fiber (in this case $SU(2)/U(1)$).

Some Details (in more familiar notation)

• Pauli considers the six-dimensional total space $M \times S^2$, where S^2 is the two-sphere on which $SO(3)$ acts in the canonical manner. He distinguishes among the diffeomorphisms (coordinate transformations) those which leave the space-time manifold M pointwise fixed and induce space-time-dependent rotations on S^2:

$$(x, y) \to [x, R(x) \cdot y]. \tag{3}$$

• P. postulates metric on $M \times S^2$ that is supposed to satisfy three assumptions $\to$ non-Abelian Kaluza-Klein ansatz:

Metric $\widehat{g}$ on the total space is constructed from a space-time metric g, the standard metric γ on S^2, and a Lie-algebra-valued 1-form,

$$A = A^a T_a \, , \; A^a = A^a_\mu dx^\mu, \tag{4}$$

on M (T_a, $a = 1, 2, 3$: standard generators of the Lie algebra of $SO(3)$) as follows: If $K^i_a \partial/\partial y^i$ are the three Killing fields on S^2, then

$$\widehat{g} = g - \gamma_{ij}[dy^i + K^i_a(y)A^a] \otimes [dy^j + K^j_a(y)A^a]. \tag{5}$$

In particular, the non-diagonal metric components are

$$\widehat{g}_{\mu i} = A^a_\mu(x)\gamma_{ij}K^j_a. \tag{6}$$

• P. determines the transformation behavior of A^a_μ under the group (1) and finds in matrix notation what he calls *"the generalization of the gauge group"*:

$$A_\mu \to R^{-1}A_\mu R + R^{-1}\partial_\mu R. \tag{7}$$

With the help of A_μ, P. defines a covariant derivative, which is used to derive *"field strengths"* by applying a generalized curl to A_μ. This is exactly the field strength that was later introduced by Yang and Mills. To our knowledge, apart from Klein's 1938 paper, it appears here for the first time. Pauli says that *"this is the true physical field, the analog of the field strength"* and he formulates what he considers to be his *"main result"*:

The vanishing of the field strength is necessary and sufficient for the $A^a_\mu(x)$ in the whole space to be transformable to zero.

• In a second letter P. also studies the dimensionally reduced Dirac equation and arrives at a mass operator that is closely related to the Dirac operator in internal space (S^2, γ). P. concludes with the statement: *"So this leads to some rather unphysical shadow particles"*.

• **Mass problem and encounter with Yang**

Mass problem was P.'s main concern; emphasized it repeatedly, most explicitly in second letter to Pais on December 6, 1953, after he had made some new calculations and had given the two seminar lectures in Zurich already mentioned. He adds to the Lagrangian what we now call the Yang-Mills term for the field strengths and says that *"one will always obtain vector mesons with rest-mass zero (and the rest-mass if at all finite, will always remain zero by all interactions with nucleons permitting the gauge group)."* To this Pauli adds: *"One could try to find other meson fields"*, and he mentions, in particular, the scalar fields which appear in the dimensional reduction of the higher-dimensional metric. In view of the Higgs mechanism this is an interesting remark.

Pauli learned about the related work of Yang and Mills in late February, 1954, during a stay in Princeton, when Yang was invited by Oppenheimer to return to Princeton and give a seminar on his joint work with Mills.

Yang:

"Soon after my seminar began, when I had written down on the blackboard $(\partial_\mu - i\epsilon B_\mu)\Psi$, Pauli asked: *What is the mass of this field B_μ?*, I said we did not know. Then I resumed my presentation, but soon Pauli asked the same question again. (...)."

In a letter to Yang shortly after Yang's Princeton seminar, Pauli repeats:

"But I was and still am disgusted and discouraged of the vector field corresponding to particles with zero rest-mass (I do not take your excuses for it with 'complications' seriously) and the difficulty with the group due to the distinction of the electromagnetic field remains."

Formally, Pauli had, however, all important equations, as he shows in detail, and he concludes the letter with the sentence:

"On the other hand you see, that your equations can easily be generalized to include the ω-space" (the internal space).